Closed similarity solutions for a class of stationary nonlinear Boltzmann-like equations

1983 J. Phys. A: Math. Gen. 16 L343

(http://iopscience.iop.org/0305-4470/16/11/001)

J. Phys. A: Math. Gen. 16 (1983) L343–L350. Printed in Great Britain

LETTER TO THE EDITOR

Closed similarity solutions for a class of stationary nonlinear Boltzmann-like equations

H Cornille†, A Gervois† and V Protopopescu‡

Service de Physique Theorique, CEN Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 24 March 1983

Abstract. We look for closed stationary solutions of nonlinear transport equations of the stochastic Boltzmann type in $(1+1)$ dimensions (velocity v and position x). These solutions, mainly written as the product of an exponential by a polynomial in $z=x^{\lambda}v^2$, have the quasi similarity property.

As a consequence of their polynomial character, they violate the positivity requirement; however, we think that they are the dominant term of infinite series solutions which might have the desired positivity property.

Calculations are greatly simplified by using sum rules analogous to conservation of mass and energy.

Over the past several years, the nonlinear Boltzmann equation has aroused a great deal of renewed interest after the finding first by Bobylev (1975) and then by Krook and Wu (1976) of some exact non-trivial solutions (hereafter called BKW) for the time-dependent homogeneous version for Maxwell molecules. Many papers followed rapidly (Tjon and Wu 1979, Ernst 1979, Barnsley and Cornille 1980, Hauge and Praestgaard 1981, Cornille and Gervois 1981), all of them dealing with various generalisations in the homogeneous case. (For further references see the review papers by Cornille and Gervois (1980) and Ernst (1981).) A next step could be the study of a one-dimensional stationary non-homogeneous problem

$$v(\partial/\partial x)f(x,v)=Q(f,f)=\iint \sigma(v,w,\theta)\,\mathrm{d}w\,\mathrm{d}\theta(f(x,w')f(x,v')-f(x,w)f(x,v)). \qquad (1)$$

It is clear that the presence of the velocity v in both the linear and nonlinear terms increases the difficulty of the problem and this is confirmed by the present analysis.

We choose the 'scattering cross section' σ in the form

$$\sigma(v,w,\theta)=|v|^p|w|^q\sigma(\theta) \qquad q,p\geqslant 0 \qquad (2)$$

which satisfies the positivity but not the symmetry in the incoming velocities (unless $p=q$) nor the detailed balance, (unless $p=q=0$; Kac's model (1956)). It seems difficult to accept on physical grounds the asymmetry in the incoming velocities. However, for the sake of mathematical completeness and because of several mathematical phenomena which might be interesting either in themselves or for other problems, we shall include these cases too in our analysis. By the way, we remark

† Chercheur CNRS.

‡ Permanent address: Sterling Chemistry Laboratory, Yale University, Connecticut, USA.

that the case $p=1$, $q=0$ leads to the closest formal analogy with the time-dependent homogeneous problem: $\partial_t \to \mathrm{sg}\, v\, \partial_x$, where $\mathrm{sg}\, v$ is the sign function $v/|v|$.

The cross section $\sigma(v, w, \theta)$ given by (2) is a stochastic one—it does not come from a mechanical law of scattering. In analogy with the Kac model (Kac 1956), we shall describe the change of the velocities in a collision by the transformation

$$v' = v\cos\theta - w\sin\theta \qquad w' = v\sin\theta + w\cos\theta \tag{3}$$

where θ should not be interpreted like an angle (for the motion remains always one-dimensional) but as a kinematical parameter. The transformation (3) does not preserve the momentum but locally preserves the mass and the energy, though mass and energy may vary throughout the vessel. We can reasonably suppose $\sigma(\theta)=\sigma(\pi-\theta)=\sigma(\pi+\theta)$. Then, decomposing f in its even and odd parts with respect to v, $f=f_+ + f_-$, we get from (1)

$$f_-(x, v) = -(v/|v|^p)(N_q(x))^{-1}(\sigma_0)^{-1}\partial_x f_+(x, v) \tag{4}$$

$$-\frac{v^2}{|v|^p}\partial_x\left(\frac{1}{N_q(x)}\frac{\partial_x f_+(x, v)}{\sigma_0}\right)$$

$$= \iint \sigma(\theta)\,\mathrm{d}\theta\,\mathrm{d}w\,|v|^p|w|^q[(f_+(x, v')f_+(x, w') - f_+(x, v)f_+(x, w)] \tag{5}$$

$$N_q(x) \equiv \int |w|^q f(x, w)\,\mathrm{d}w = \int |w|^q f_+(x, w)\,\mathrm{d}w. \tag{6}$$

The moment $\sigma_0 = \int \sigma(\theta)\,\mathrm{d}\theta$ appears only with $N_q(x)$. It is finite if $\sigma(\theta)$ is not too singular and in the following, we always assume $\sigma_0 = 1$. Throughout the paper we assume $N_q(x) \not\equiv 0$, so that f_- is unambiguously determined from f_+; then equation (5) is a closed equation for f_+.

The particular role played by the BKW solutions in the homogeneous case suggests to look for solutions of the same type, namely

$$f(x, v) = A(x)\exp(-b(x)v^2) \times \text{polynomial in } \sqrt{b}v \tag{7}$$

where two main ideas are included.

(i) The solution has a Maxwell-like factor; this factor appears in the resolution of $Q(f, f)=0$, but it is not an ordinary local equilibrium solution as the momentum is not conserved. Moreover, the meaning cannot be the same as in the t-dependent case, where we looked for solutions tending to $\exp(-v^2/2)$, and expression (7) cannot be thought of as a perturbation, as we can have stationary solutions far away from the equilibrium.

(ii) A similarity transformation is used, contracting the two independent variables (x, v) into a single new one $z=b(x)v^2$. When $A(x)$ in (7) is a constant, we have *pure similarity* solutions; in the general case, we shall speak of *quasi-similarity*. Because of (7) and the similarity condition, we choose

$$f_+(x, v) = \exp(-b(x)v^2)[P(x) + Q(x)v^2 + R(x)v^4 + S(x)v^6 + \ldots + V(x)v^{2N}]. \tag{8}$$

The requirement of polynomial solutions compatible with equation (5) leaves us with only three possibilities: $p=2$, $N=1$; $p=1$, $N=2$ and $p=0$, $N=3$. Further, the same requirement implies that, when either $v \to +\infty$ or $-\infty$, necessarily f_- and $f + f_+ + f_-$

become negative; the reason is that both f_- and f are v polynomials of higher degree than f_+ with an odd dominant term. In fact it appears from our study that f_+ itself can have negative parts. This suggests that the form (8) might actually describe only a germ of a solution which is in fact an infinite series and eventually displays the desired positivity property.

From the equations above, we derive sum rules analogous to conservation of mass and energy in the time-dependent Boltzmann-like equations and which appear very powerful in the present study. By multiplying (5) $|v|^{n-p}$ for any n ($n > 2p-3$) and integrating over v, we can write

$$-\partial_x\left(\frac{1}{N_q(x)}\partial_x N_{2-2p+n}(x)\right)=\int\sigma(\theta)\,\mathrm{d}\theta\iint \mathrm{d}v\,\mathrm{d}w\,f_+(v)f_+(w)(|v'|^n|w'|^q-|v|^n|w|^q)$$

and the RHS disappears when both $n=q=0$ or when either $n=0$, $q=2$ or $n=2$, $q=0$. For $p=2$ and $q=0$ we thus get one sum rule ($n=2$); for $p=1$ and $p=0$, we have three sum rules for $q=0$ ($n=0,2$) and $q=2$ ($n=0$).

We come now to the result of our study. Things become more involved when p decreases from 2 to 0. One reason for this complexity is that the degree of the polynonial in f_+ increases. Another reason is the appearance of the first moments σ_{2n} of the cross section $\sigma(\theta)$ of equation (2)

$$\sigma_{2n}=\int_0^{2\pi}\mathrm{d}\theta\,\sigma(\theta)(\sin\theta\cos\theta)^{2n}.$$

In determining f_+, the cross sections appear for $p=2$ only through an inessential multiplicative factor σ_2, like in the BKW solution, whereas for $p=1$ (resp $p=0$) the ratio σ_2/σ_4 (resp σ_2/σ_6, σ_4/σ_6) enters into the formalism; only discrete values of these ratios correspond to true solutions. Besides, positivity constraints for $\sigma(\theta)$ require $\sigma_2/\sigma_4>1$ (resp $\sigma_2/\sigma_6>\sigma_4/\sigma_6>1$).

For $p=2$, the whole calculation may be done analytically for any q. For $p=1$ and $p=0$, we found solutions numerically, except when $q=0$ or 2 where the sum rules provide analytical results. For $p=1$, in supplement to the sum rule cases $q=0,2$, we have performed a complete study for any q which interpolates the $q=0$ and 2 results. On the contrary, for $p=0$, we restricted to the analytical cases $q=0,2$ which presumably are limiting values in a complete q study. We retain as available solutions those for which (i) f is a real function (not necessarily everywhere positive) with finite positive density $N_0(x)=\int f(x,v)\,\mathrm{d}v$, (ii) $\sigma(\theta)$ is positive.

Now, choosing for f_+ the expression (8) and substituting into (6)–(7), we get N_q and a polynomial in v^2 (whose maximum degree is 4, 8 or 12 for $p=2,1,0$ respectively) with q-dependent coefficient which must be identically zero for the different powers of v^2.

The consistency of these equations together with the similarity conditions give $Q\sim Pb$, $R\sim Pb^2$, $S\sim Pb^3$, $N_q\sim Pb^{-(q+1)/2}$.

Anticipating a main result, i.e. $b(x)$ is a power law $b(x)=b_0(x-x_0)^\lambda$ with $x>x_0$, and setting $x_0=0$ because of translational invariance, we have

$$P=P_0x^{\lambda(p+q)/2-1}\qquad Q=Q_0x^{\lambda(p+q+2)/2-1}\qquad R=R_0x^{\lambda(p+q+4)/2-1}$$
$$S=S_0x^{\lambda(p+q+6)/2-1}\qquad N_s=N_{s_0}x^{\lambda(p+q-s-1)/2-1}.\tag{9}$$

where P_0, Q_0, R_0, S_0, N_{s_0} are constant.

These expressions are written generally; for $p=1$, we must set $S\equiv 0$ and for $p=2$, $R\equiv S\equiv O$. They imply that equation (1) is invariant under the change $x\to\alpha^{\mu}x$, $v\to\alpha^{\nu}v$ provided $\lambda=-2\nu/\mu$. Functions f_+ and f_- are rewritten

$$f_+(x,v)=x^{\lambda(p+q)/2-1}F_+(z) \tag{10a}$$

$$f_-(x,v)=-N_{q_0}^{-1}\,\mathrm{sg}\,v\,x^{\lambda(p+q)/2-1}|z|^{(1-p)/2}F_-(z) \tag{10b}$$

$$z=x^{\lambda}v^2 \qquad F_+(0)=P_0 \qquad F_-(0)=[\lambda(p+q)/2-1]P_0 \tag{10c}$$

where $F_\pm$ are functions of z only (product of an exponential and a polynomial) and $F_-(0)$ is zero in the pure similarity case $\lambda=2/(p+q)$.

This kind of x dependence introduces unpleasant features at small or large x. We simply verify *a posteriori* that: (i) For $\lambda<0$ and $\lambda>\max(2/(q+1),\,2/(q+p))$ the limit of f when $x\to 0^+$ is zero whereas when $x\to\infty$, $f\to 0$ for any λ. The behaviour of the density $N_0(x)$ depends on the effective values of p, q, λ. In general, N_0 diverges like powers of x. (ii) For $0<\lambda<\min(4/(q+1),\,2/(q+p))$, we have a breakdown both of f and the derivative $\partial f/\partial x$ at $x=0$. (iii) For intermediate regions, different regimes occur.

The results are somewhat different for the pure similarity case as $F_-(0)=0$. We find three such solutions ($p=q=2$; $p=1$, $q=0.08$ and $q=2.002$). For all of them f is well defined for $x=0$ and $f_+|_{x=0}=$ constant, $f_-|_{x=0}=0$ (specular reflection). For one of them ($p=1$, $q=0.08$), $\partial f/\partial x|_{x\to 0^+}\to 0$; for the other ones, the derivative is infinite.

Case $p=2$ ($R\equiv S\equiv O$)

Beside the non-negative powers of v^2, an extra term appears in this case only, corresponding to v^{-2} (equation [−2] in table 1). Due to the first and last equations [−2], [4] in table 1, two possibilities occur. In the first one $P_x\not\equiv 0$, $(b_x)^2b^{(q-1)/2}=$ constant, $P\sim b^{1/2}$. In the second one $P\equiv$ constant, we find $(b_x)^2b^q=$ const. In both cases, the x dependences are of the *power* type $b(x)=b_0(x-x_0)^{\lambda}$, $\lambda=2/(q+1)$, $N_q\sim(x-x_0)^{(1/(q+1))-1}$ and $\lambda=2/(q+2)$, $N_q\sim(x-x_0)^{(1/(q+2))-1}$ respectively with $x>x_0$ and the corresponding ones for P, Q. At this stage, only the parameter q and the constants in front of the power laws are to be determined. For this, we use the two

Table 1. Equations for $f_+(x,v)$ when $p=2$ and solutions for $q=14$ and $q=2$.

[−2] $P_{xx}N_q=N_{qx}P_x$ [0] $(Pb_{xx}+2b_xP_x-Q_{xx})+(Q_x-b_xP)\dfrac{N_{qx}}{N_q}=\dfrac{\Gamma((q+5)/2)}{b^{(q+5)/2}}\sigma_2N_qQ^2$

[2] $2Q_xb_x+Qb_{xx}-P(b_x)^2-Qb_x\dfrac{N_{qx}}{N_q}=-6\Gamma((q+3)/2)\dfrac{\sigma_2N_qQ^2}{b^{(q+3)/2}}$

[4] $-(b_x)^2b(q+1)/2=\Gamma((q+1)/2)\sigma_2N_qQ$

$N_q=\Gamma((q+1)/2)b^{(q+3)/2}[bP+\dfrac{(q+1)}{2}Q]$

$q=14$ $f_+=P_0x^{1/15}\,\mathrm{e}^{-z}(1-\frac{2}{83}z)$, $z=b_0x^{2/15}v^2$, $83b_0^8=P_0\sqrt{2}\Gamma(\frac{15}{2})15\sqrt{34\sigma_2}$

$f_-=-(P_0b_0^{1/2}/15N_{14,0})(\mathrm{sg}\,v)\,x^{1/15}\,\mathrm{e}^{-z}\,z^{-1/2}(1-\frac{172}{83}z+\frac{4}{83}z^2)$

$q=2$ $f_+=Q_0\,\mathrm{e}^{-z}(-\frac{13}{2}+z)$, $z=b_0x^{1/2}v^2$, $b_0^4=5\pi\sigma_2Q_0^2$

$f_-=(b_0^2/5\sqrt{\pi})(\mathrm{sg}\,v)\,\mathrm{e}^{-z}\sqrt{z}(\frac{15}{2}-z)$

remaining equations ($2n = 0$ and 2) of table 1. We are left with only three discrete q values: $q = 0, 2, 14$.

For $q = 0$, the sum rules read $\partial_{x^2}^2(\ln N_0) = 0$. In the first case, N_0 is a constant and the sum rule is automatically fulfilled but it turns out that the final distribution f_+ is complex and we disregard this case. In the second case $N = N_{00}x^{-1/2}$ and the sum rule cannot be satisfied ($N_0(x) \not\equiv 0$). Solutions for $q = 14$ and $q = 2$ are acceptable and the last one satisfies pure similarity. As discussed above, they have negative parts but in z regions where they are relatively small.

Case $p = 1$ ($S \equiv O$)

General equations for functions P, Q, R, b are given in table 2 (equation [0] to [8]). Equations for N_q and [8] together with the scaling condition give $P \approx b_x b^{(q-1)/2}$, $N_q \sim b_x/b$. Assuming for b a power law $b = b_0 x^{\lambda}$, we deduce the scaling laws equation (9) and a system of equations for the constants b_0, P_0, Q_0, R_0, N_{q0} in terms of parameters λ, $\lambda^2\sigma_2/\sigma_4$ and q. By solving the equations, we can eliminate P_0, $Q_0, \ldots$ and with the help of the computer we determine *for every* q a finite number of possible $(\lambda, \sigma_2/\sigma_4)$ which lead to real distributions. The results for λ are plotted in figure 1 for positive q. Two pure similarity solutions exist, and the corresponding λ is obtained as the intersection of $\lambda = \lambda(q)$ with the critical curve $\lambda = 2/(q+1)$ (see full circles on figure 1); we find $q \sim 2.002$ ($\sigma_2/\sigma_4 \sim 5.8$) and $q \sim 0.08$($\sigma_2/\sigma_4 \sim 8$). We recall that they satisfy the specular reflection condition $f_-(x = 0, v) = 0$. The symmetrical case $p = q = 1$ provides two solutions $\lambda = 0.297$, $\sigma_2/\sigma_4 = 8.068$ and $\lambda = -0.144$, $\sigma_2/\sigma_4 = 4.10$. Once the pairs $(\lambda, \sigma_2/\sigma_4)$ are obtained, we determine all remaining constants but b_0 and $\sigma(\theta)$ which remain almost arbitrary. The only constraint is positivity and the knowledge of the ratio σ_2/σ_4. Functions f_+ and f_- are given by (10). The only remaining problem is the determination of classes of constants $\sqrt{\sigma_2}$ consistent with the constraints on $\sigma(\theta)$. (Schwartz's inequality $\sigma_2/\sigma_0 < \sigma_4/\sigma_2$

Table 2. Equations for $f_+(x, v)$ when $p = 1$ and similarity functions $F_\pm(z)$ from which we build f_+ and f_-.

[0] $P_x(N_{qx}/N_q) - P_{xx}$

$$= [\Gamma((q+5)/2)/b^{(q+9)/2}]N_q[\sigma_2 b^2(Q^2 - 2PR) + \sigma_2\tfrac{1}{2}(q+5)bQR + \sigma_4\tfrac{1}{4}(q+5)(q+7)R^2]$$

[2] $(Q_x - Pb_x)(N_{qx}/N_q) + 2P_x b_x + Pb_{xx} - Q_{xx}$

$$= [\Gamma((q+3)/2)N_q/b^{(q+7)/2}][-6\sigma_2 b^2(Q^2 - 2PR) - \tfrac{5}{2}(q+3)\sigma_2 bQR + \tfrac{1}{2}(q+3)(q+5)(3\sigma_2 - 14\sigma_4)R^2]$$

[4] $(R_x - Qb_x)(N_{qx}/N_q) + 2Q_x b_x + Qb_{xx} - R_{xx} - P(b_x)^2$

$$= [\Gamma((q+1)/2)N_q/b^{(q+5)/2}][\sigma_2 b^2(Q^2 - 2PR) - \tfrac{5}{2}(q+1)\sigma_2 bQR + \tfrac{5}{2}(q+1)(q+3)(-2\sigma_2 + 7\sigma_4)R^2]$$

[6] $-Rb_x(N_{qx}/N_q) + 2R_x b_x + Rb_{xx} - C(b_x)^2$

$$= [\Gamma((q+1)/2)N_q/b^{(q+3)/2}][\sigma_2 bQR + (q+1)(3\sigma_2 - 14\sigma_4)R^2]$$

[8] $-b^{(q+1)/2}(b_x)^2 = \Gamma(\tfrac{1}{2}(q+1))\sigma_4 N_q R$

$$b^{(q+5)/2}N_q = \Gamma(\tfrac{1}{2}(q+1))[b^2P + \tfrac{1}{2}(q+1)bQ + \tfrac{1}{4}(q+n)\tfrac{1}{4}(q+1)(q+3)R]$$

$$F_+(z)\sqrt{-(\sigma/\mathcal{N})} \times (\mathrm{e}^{-z}/R_0)(P_0 b_0^2 + Q_0 b_0 z + R_0 z^2)$$

$$F_-(z) = -(-\mathcal{N}\sigma)^{-1/2}[(1-\lambda) - \lambda z\,\partial_z]F_+$$

$$= [b_0^2 P_0 + \tfrac{1}{2}(q+1)b_0 Q_0 + \tfrac{1}{4}(q+1)(q+3)R_0]/R_0, \qquad \sigma = \lambda^2\sigma_2/\sigma_4$$

gives a limitation.) As an illustration, we represent in figure 2 the symmetrical case $q = 1$ with possible $\lambda = -0.144$ and $\sqrt{\sigma_2} = 0.20$ corresponding to the allowed $\sigma(\theta) = [1 + 0.41 (\sin\theta \cos\theta)^{-0.8}] \times 0.0446$.

We would like to emphasise the role of $q = 0$ and $q = 2$ which clearly appear in figure 1 as limiting cases (the cross in figure 1 marks the end points for $q = 0$ or $q = 2$).

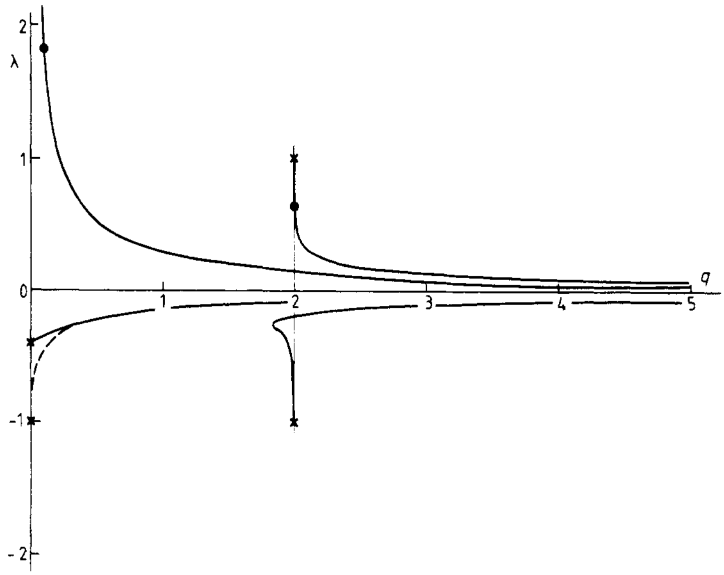

Figure 1. Plot of λ against q for $p = 1$, $q \geq 0$.

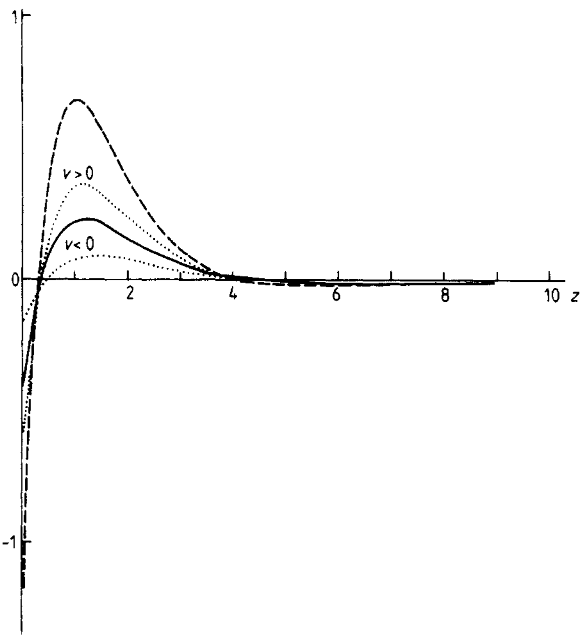

Figure 2. Plot for $p = 1$, $q = 1$ and $\lambda = -0.144$ for the reduced functions $F_+(z)/\sqrt{\sigma_2}$ (full curve) and $F_-(z)$ (broken curve) against z. The total function $x^{1-\lambda} f(x, v)$ is proportional to $F_+(z) + \mathrm{sg}\, v\, F_-(z)$ (dotted lines for $v > 0$ and $v < 0$; $\sigma_2 = 0.0387$ $\sigma_2/\sigma_4 = 4.100$).

They can be completely studied analytically, thanks to the simplification due to the sum rules that we discuss now.

For $q=2$, the previous condition $N_q(x)\not\equiv 0$ eliminates the limit $(q\to 2)$ solutions $\lambda=-\frac{2}{25}$ and $\lambda=-1$. Now, the unique sum rule reads $\partial_x(N_2^{-1}\,\partial_x N_0)=0$ or $(\lambda-1)^2 N_{00}=0$. If $N_{00}=0$, one of the equations is replaced by the sum rule $P_0 b_0^2+\frac{1}{2}Q_0 b_0+\frac{3}{4}R_0=0$ which simplifies the formalism. We find $\lambda=0.136$, $\sigma_2/\sigma_4=8.93$ and $\lambda=-0.203$, $\sigma_2/\sigma_4=4.03$ which fulfil all our criteria except positive density. If $\lambda=1$, the remaining unknown parameter $\sigma=\sigma_2/\sigma_4$ is a solution of the fourth-order equation $360\sigma^4-2442\sigma^3+3553\sigma^2-6512\sigma-62=0$. We find $\sigma\sim 5.59>1$.

For $q=0$ the two sum rules are written $\partial^2_{x^2}\ln N_0=0$ and $\partial_x(N_0^{-1}\partial_x N_2)=0$. We recall that $N_0(x)=N_{00}x^{-1}$ so that the first sum rule cannot be satisfied. As $N_2(x)=N_{20}x^{-(\lambda+1)}$, the second sum rule is rewritten $N_{20}(\lambda+1)^2/N_{00}=0$ leading either to $\lambda=-1$, $\sigma=4\pm 3\sqrt{2}$, or to $N_{20}\propto P_0 b_0^2+\frac{3}{2}Q_0 b_0+\frac{15}{4}R_0=0$; $N_{20}=0$ corresponds to $\lambda=-\frac{2}{5}$, $\sigma=1$. Though these solutions must be rejected in our formalism $(N_0(x)\equiv 0)$, it is remarkable that they are the limit of the $q\to 0^+$ solutions, and we emphasise that some of these $q\to 0$ solutions are acceptable (see figure 1, full curves).

Case $p=0$

We give here only the higher-order term, the coefficient of v^{12}

$$-(b_x)^2 b^{(q+1)/2}=\sigma_6\Gamma((q+1)/2)N_q S.$$

With similarity constraints (9) we get $P\sim b_x b^{q/2-1}$, $N_q\sim b_x b^{-3/2}$. The constants P_0, Q_0, R_0, S_0, N_{s_0}, b_0 are to be determined as functions of the parameters λ, $\lambda^2\sigma_2/\sigma_6$, $\lambda^2\sigma_4/\sigma_6$ and q. The complete numerical analysis can be performed for any q value but for simplicity, we restrict to the analytical cases $q=0$ and $q=2$ where we take great advantage of the sum rules. For $q=0$, the first sum rule gives $\partial_x(N_0^{-1}\,\partial_x N_2)=0$ or $(3\lambda/2+1)$ $(\lambda+1)N_{20}=0$, whereas the second one $\partial_x(N_0^{-1}\,\partial_x N_4)=0$ leads to $(\frac{5}{2}\lambda+1)(2\lambda+1)N_{40}=0$. We have two cases:

(i) either N_{20} and $N_{40}\neq 0$; we find $\lambda=-\frac{1}{3}$, $-\frac{2}{7}$, $-\frac{1}{4}$, $-\frac{2}{9}$, but they do not satisfy the positivity of $\sigma(\theta)$;

(ii) or either $N_{20}=0$, $\lambda=-\frac{2}{5}$ and $\lambda=-\frac{1}{2}$ or $N_{40}=0$, $\lambda=-\frac{2}{3}$ and $\lambda=-1$; again this case does not lead to solutions with positive $\sigma(\theta)$. In conclusion, if they exist, closed solutions for the Kac model are not of this type.

For $q=2$, the unique sum rule is $\partial^2_{x^2}\ln N_2=0$ or $N_2(x)=$ constant and corresponds to macroscopic conservation of energy. As $N_2(x)=N_{20}x^{-(\lambda/2+1)}$ and $N_2(x)\not\equiv 0$, we must have $\lambda=-2$. We find two acceptable (although not always positive) solutions corresponding to the parameter values

$$\sigma_2/\sigma_6=32.9 \qquad \sigma_4/\sigma_6=5.07 \tag{11a}$$

$$\sigma_2/\sigma_6=29.27 \qquad \sigma_4/\sigma_6=6.36 \tag{11b}$$

with density $N_0(x)=N_{00}x^{-2}(N_{00}>0)$ and scaling variable $z=b_0 v^2/x^2$.

As for $p=1$, using (10), we get

$$F_+(z)=(2b_0\,\mathrm{e}^{-z}/(-\mathcal{N}\sigma_6)^{1/2}S_0\Gamma(\tfrac{3}{2}))(P_0 b_0^3+Q_0 b_0^2 z+R_0 b_0 z^2+S_0 z^3)$$

$$F_-(z)=(-b_0/2\sqrt{-\mathcal{N}}\Gamma(\tfrac{3}{2}))(3+2z\,\partial_z)F_+ \qquad \text{and } \mathcal{N}=(P_0 b_0^3+\tfrac{3}{2}Q_0 b_0^2+\tfrac{15}{4}R_0 b_0+\tfrac{105}{8}S_0)/S_0.$$

For the remaining unknown σ_6, the Schwartz inequality gives $\sigma_6/\sigma_0\leqslant(\sigma_4/\sigma_6)(\sigma_2/\sigma_6)^{-2}$ and we explicitly build possible $\sigma(\theta)$. For instance, solution (11a) with $\sigma(\theta)=0.311\times[1-4\sin^2\theta\cos^2\theta+3.81\,(\sin\theta\cos\theta)^{6.6}]$ gives $\sigma_6=0.001\,47$. In figure 3, we have plotted the reduced function proportional to $x^3 f(x,v)$, corresponding to this case. We recall that, like for $p = 2$ and $p = 1$, only b_0 is undetermined and yields a multiplicative overall constant coming from the invariance of the equations to the transformations $(x, f) \to (\rho x, \rho f)$.

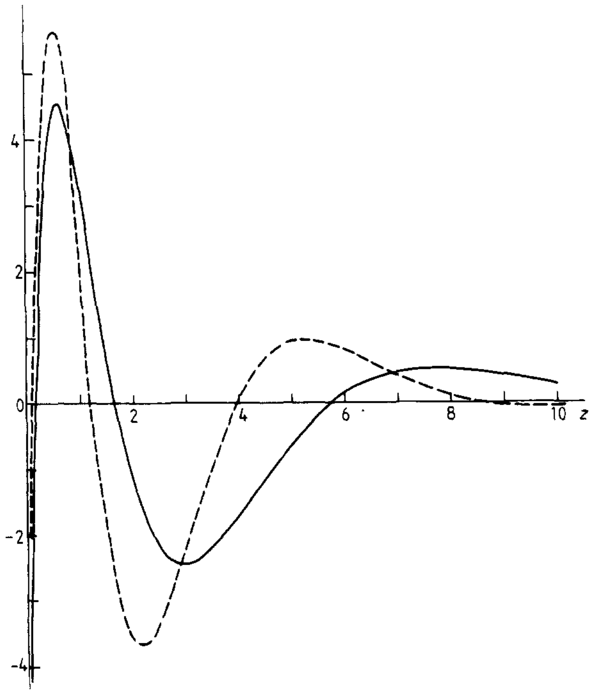

Figure 3. Plot of $x^3 f(x, v)$ for $v > 0$ (full curve) and $v < 0$ (broken curve) in terms of $z = b_0 v^2/x^2$ for $p = 0$, $q = 2$ and $\lambda = -2$ (solution 11*a*). $\bar{\sigma}_2 = 32.90$, $\bar{\sigma}_4 = 5.067$, $\sigma_6 = 0.001\,49$.

One of the authors (VP) acknowledges Professor T Keyes for his interest in this work.

Note added in proof. We stress that the solutions we get in this letter are obtained from the sourceless equation only and without any additional conditions at the 'critical' points $x = 0, \infty$. Moreover, we always restricted to the halfspace $x > 0$. If a similar study was carried out for $x < 0$, it would only remain to investigate the physical meaning of a gas on the whole x axis with a 'barrier' at $x = 0$.

References

Barnsley M and Cornille H 1980 *J. Math. Phys.* **21** 1176
Bobylev V 1975 *Dokl. Akad. Nauk* **225** 1296
—— 1976 *Sov. Phys. Dokl.* **20** 823
Cornille H and Gervois A 1980 *Problèmes inverses—Inverse problems* ed P C Sabatier (Paris : Editions du CNRS) p 27
—— 1981 *J. Stat. Phys.* **26** 181
Ernst M H 1979 *Phys. Lett.* **69**A 390
—— 1981 *Phys. Rep.* **78** 1
Hauge E H and Praestgaard E 1981 *J. Stat. Phys.* **24** 21
Kac M 1956 *Proc. 3rd Berkeley Symp. on Math. Stat. and Prob.* vol 3 (Berkeley: University of California Press) p 171
Krook M and Wu T T 1976 *Phys. Rev. Lett.* **16** 1107
Tjon J A and Wu T T 1979 *Phys. Rev.* A **19** 883